A GOALS-BASED THEORY OF UTILITY

Franklin J. Parker, CFA
Chief Investment Officer
Bright Wealth Management, LLC
Dallas, Texas

franklinjamesparker@gmail.com

Original Manuscript: *May 2019*
This Version, Accepted Manuscript: *October 2019*
Accepted: *November 2019*

This article is an Accepted Manuscript of an article published by Taylor & Francis in *The Journal of Behavioral Finance*, and is available online at https://doi.org/10.1080/15427560.2020.1716359. Recommended citation:

Parker, F.J. (2020). "A Goals-Based Theory of Utility" *Journal of Behavioral Finance*, forthcoming.

Abstract

Theoretical frameworks to date have prescribed how an investor should allocate wealth within mental accounts. However, there is no fully cohesive solution to prescribe how an investor should rationally allocate resources *across* mental accounts. It is the aim of this discussion to fill that theoretical gap. We present a framework which can be used to rationally allocate resources both within and across mental accounts or goals. We then compare and contrast this method with mean-variance optimization and behavioral portfolio theory, showing that both are stochastically dominated by the goals-based utility framework. In further analysis of empirical validity, we discuss the Samuelson Paradox, Friedman-Savage puzzle, and probability weighting functions.

Keywords: mental accounting, goals-based investing, portfolio theory, utility theory

JEL Codes: C61, D01, D14, G11, G40

Author Acknowledgements: *I am deeply grateful to the peer-reviewers and the managing editor of the Journal of Behavioral Finance who legitimately improved this discussion. In addition, though there are many people who have consistently made this work possible, I am particularly indebted to Patrick Foster. It was in conversation with him, many years ago, that this work found its genesis.*

Nobel-laureate Robert Shiller has stated that "Finance is not merely about making money. It is about achieving our deep goals and protecting the fruits of our labor."[1] Whether individuals or institutions, the act of investing is not done in the abstract, there is some ultimate aim in mind. Investing is, at its foundation, goals-based. Only recently has this fact been given an accounting in theoretical frameworks. Thaler (1985) presented a framework where individuals mentally divide their portfolios into "buckets," a concept he called mental accounting. Each mental account has a separate objective and thus a separate risk tolerance—a framework which has been used to explain several behavioral anomalies, including the equity premium puzzle (Bernartzi and Thaler, 1995).

Shefrin and Statman (2000) took Thaler's work to its logical conclusion with their behavioral portfolio theory (BPT). In BPT, risk is redefined as the probability of falling short of the account's objective. The economic agent seeks a trade-off between higher expected wealth and lower probabilities of achievement—a firm move away from the variance-is-risk framework of Markowitz (1952a, 1959). Following on from their initial work, Das, et al (2010), sought to merge the risk-is-goal-failure framework of BPT with the mean-variance framework; primarily, their work shows that mean-variance optimization and goal achievement probability maximization are mathematically synonymous in most cases.

Despite this theoretical development, there still exists a rather conspicuous hole. The current state of the theory prescribes how an agent should allocate wealth *within* a mental account (or goal), carrying the implicit assumption that the agent has already allocated across mental accounts. There is currently no solution to prescribe how to rationally allocate capital *across* multiple mental accounts. It is this gap in the literature which this discussion aims to fill.

And this is not some theoretical question. Individuals and institutions regularly face multiple and often mutually exclusive goals over varying time horizons. For example, an individual who wishes to retire in twenty years, send kids to college in ten years, and buy a vacation home in three years, faces an impasse. Most likely retirement is the most important goal, but can a vacation home be funded with an acceptable loss of achievement probability for retirement and college savings? A rational and practical approach is needed for allocating scarce resources across multiple, often mutually exclusive goals.

We should here acknowledge that BPT did offer some advice toward a solution. Their heuristic has agents allocate wealth in a "layered" fashion, prioritizing goals from high to low. The subsequent goals-based frameworks, such as those proposed by Chhabra (2005) and Nevins (2004), are rooted in this "layered" allocation. That said, this solution does not offer a relative valuation of each mental account to another, nor is there a clear substitution methodology. Further, in the synthesis work of Das, et al (2010), there is an implicit assumption that the investor has already rationally subdivided resources into various mental accounts—effectively eliminating the solution offered in BPT. The approach presented herein attempts to push the work of original BPT forward by constructing a more concrete, rational, and practical utility function, thereby replacing the implicit assumptions of former frameworks with firmer theory.

A quick note before proceeding. The framework presented in this discussion is rooted in mental accounting, leading to the apparent critique that mental accounting, on the whole, is not rational. There are, however, two uses of the term "mental accounting." There is (a) the behavioral bias where individuals do not treat money as fungible, and (b) the allocation of wealth to different accounts, each dedicated to a specific goal. We are unconcerned with the former, indeed this framework will treat money as fungible (expecting a rebalance across goals). The latter was rebutted by Brunel (2006) who shows that, so long as each mental accounting portfolio lies on the mean-variance frontier, the aggregate portfolio will also lie on the frontier; a result further confirmed by Das, et al (2010). Thus, when short selling is allowed, the division of wealth into multiple accounts is not inefficient so long as each account is optimized within the context of the other accounts.

In addition, there exist real-world frictions which force a physical manifestation of the mental accounting framework. Many tax jurisdictions, for example, allow for the contribution of capital to tax-advantaged and locked-up account types, like retirement and education accounts. There currently does not exist a cohesive framework to determine how capital should be allocated across these, decidedly non-mental, accounts.

Also of interest, as we will show, the goals-based approach generates a distinct optimization procedure and investment preferences from traditional mean-variance (MV) portfolios. Lottery-like (high variance, negative expected value) investments, are nigh eliminated from the MV universe. Of course, this has spawned such utility puzzles as presented by Friedman and Savage (1948). In the procedure presented here, however, such assets can play an important role.

[1] "Q&A with Robert J. Shiller" from Princeton University Press, https://press.princeton.edu/interviews/qa-9652 retrieved April 22, 2019.

Our discussion is organized as follows: we first derive a goals-based theory of utility. We then consider a concrete example to illustrate the theoretical application, and compare the framework with that of MV and BPT. We show that both are stochastically dominated by the goals-based model. Finally, we view several utility puzzles through the lens of this model, addressing the Samuelson paradox, the Friedman Savage Puzzle, and probability weighting functions. We close with some discussion.

1. Utility of Goals

We propose that economic agents populate a goal space. We make no claim how this is done, only that it is. Further, we propose that agents assign values to each goal in the space, creating a hierarchy of preference, and we assert, axiomatically, that the agent prefers higher probabilities of achievement to lower. Finally, we propose that this hierarchy of preference follows von-Neumann and Morganstern's (1944, hereafter VNM) axioms of choice. That is, preferences are complete, transitive, continuous, and independent across the goal space. Given these starting assumptions, it is possible to construct a rational framework for the allocation of scarce resources.

1.1 Toward a Theory of Goals

Let the agent's goal space be $G\colon\{A, B, \ldots, N\} \in G$, where $A, B, \ldots, N$ are an N-number of separate goals which compete for the agent's resource pool, ϖ. We define a goal as a two-variable vector: $A = (W_A, t_A)$, where W_A is the wealth required to fund the goal and t_A is the time horizon within which the goal must be accomplished. As mentioned, we expect VNM's axioms of choice to apply to preferences across G. Through the axiom of completeness, G can be rank-ordered, $A \succcurlyeq B \succcurlyeq \cdots \succcurlyeq N$. For mathematical convenience, let there exist some function, $v(\cdot)\colon A \succcurlyeq B \Leftrightarrow v(A) \geq v(B)$. It follows, $\exists p\colon v(B) = v(A) \cdot p$. Using a certainty-equivalent method, the value p can be attained. Let $v(A) = a\colon a \in \mathbb{R}^+$ (for convenience, we set $a = 1$, though any positive real number will do[2]). The agent is then offered a prospect: the achievement of goal B with certainty, or the achievement of goal A with probability p. By systematically varying the value of p, the point of indifference is found such that $v(B) = v(A) \cdot p$, or as $v(A) = a$, we can equivalently say $v(B) = a \cdot p$. This procedure is repeated across the goal space and mapping the value ratios one-to-one to G, $\{p, q, \ldots, n\} \rightarrow \{A, B, \ldots, N\}$, we can describe the total value of the goal space, $v(G)$, as

$$v(G) = a + ap + apq + \cdots + apq \ldots n. \tag{1}$$

Functionally, the value ratios are the relative valuations of one goal for another, and they allow for the substitution of one goal for another.

In a setting without certainty, an agent dedicates initial resources, w, to the achievement of a goal. It is implied that $w < W$, otherwise the goal would be achieved and the setting would be certain. There must exist some function of the goal's variable vector and the initial resource dedication which describes the probability of goal achievement given the initial inputs: $\exists \phi(w, W, t) = \Pr[w_t \geq W]$. Recall that ϖ is the initial resource pool from which an agent can draw, thus $\exists \vartheta\colon w = \varpi\vartheta \Rightarrow \phi(\varpi\vartheta, W, t)$.

We know from Markowitz (1959, p. 242) that the weighted sum formulation $u(G) = \sum_i v(i)p(i)$ is a valid solution to VNM's axioms of choice, where i is the index of goals and p is the probability of achievement for goal i. From above, then, we can substitute $p(i) = \phi(\varpi\vartheta, W, t)$. It is therefore the objective of the agent to maximize utility by varying the initial resource allocation to each goal:

$$\max_{\vartheta} \sum_{i}^{N} v(i)\phi(\varpi\vartheta_i, W_i, t_i)\,. \tag{2}$$

A keen observer will note that equation (2) is myopic. Properly speaking, the agent must maximize inter-temporal utility in continuous time. From a practical perspective, however, this myopia is overcome by re-optimizing in discrete time, such as annually or quarterly. For purposes of this discussion, we will consider a goal accomplished when $w = W$. Upon vesting, the investor converts risk capital into an actuarially-fair annuity, the resource pool reduces accordingly, the goal drops from the goal space, and the investor re-optimizes. Any excess wealth is redirected toward other goals in the space.

The application to an investment setting is straightforward. The agent allocates to each goal, then allocates across investments in an effort to maximize the probability of achieving that goal.

We understand the annual return required to achieve a goal as $r(\vartheta) = (W/\varpi\vartheta)^{1/t} - 1$. With an assumption of normality, we can use a cumulative distribution function to describe the probability of attaining the return required by a goal, as a function of resource allocation:

[2] Note that $a \in \mathbb{R}^-$ yields a negative utility for achievement.

$$\phi(\varpi\vartheta, W, t) = 1 - \Phi\left[\left(\frac{W}{\varpi\vartheta}\right)^{\frac{1}{t}} - 1; m, s\right]. \quad (3)$$

$\Phi(\cdot)$ is the lower-tail cumulative distribution function, m is the expected portfolio return, and s is the expected standard deviation of the portfolio, all defined as functions of the weights to investments in the investment universe:

$$m = \sum_i \omega_i R_i \,, \quad (4)$$

$$s^2 = \sum_i \omega_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} \omega_i \omega_j \rho_{i,j} \sigma_i \sigma_j \,, \quad (5)$$

where $\rho_{i,j}$ is the correlation of asset i to asset j, σ_i^2 is the variance of asset i, and ω_i is the weight of asset i, and R_i is the expected return of the i^{th} asset. The agent, therefore, is left to maximize the utility function by varying the allocation of the initial resource pool to each goal *and* the weight of each asset in the investment universe:

$$\max_{\vartheta, \omega} \sum_i^N v(i) \phi(\varpi\vartheta_i, W_i, t_i) \,. \quad (6)$$

When operating outside an investment setting (or in a sufficiently non-normal investment setting), equations (3) through (5) can be substituted for whatever function most accurately describes the setting. For purposes of this discussion, we will use both the logistic and Gaussian form, though we recognize that an assumption of normality is more for tractability than for real-world accuracy.

1.2 Practical Implementation—A Concrete Example

In an effort to illustrate the theory discussed, we here present a case study. We assume Gaussian normal distributions throughout this example, though we acknowledge this is more for illustrative ease than for real-world accuracy. Suppose an economic agent presents with the following characteristics:

- Three goals, $G = \{A, B, C\}$, with funding requirements and time horizons of:
  - $2,000,000 needed in 25 years, $A = (\$2{,}000{,}000; 25)$;
  - $273,000 needed in 9 years, $B = (\$273{,}000; 9)$;
  - $43,000 needed in 7 years, $C = (\$43{,}000; 7)$;
- It is determined that the value ratios are $v(A) = a = 1$, $b = 0.45$ and $c = 0.23$;
- A total resource pool of $\varpi = \$972{,}000$, split between present value of human capital and investible dollars (for exposition on the human capital problem, we direct the readers to authors such as Ibbotson, Milevsky, Chen, and Zhu, 2005, or Baptista, 2012, who offers a human capital model in the presence of background risk);
- There exists a universe of three potential assets with the characteristics illustrated in Table 1.

TABLE 1 Asset Characteristics

	r	σ	*Correlations*	Asset 1	Asset 2	Asset 3
Asset 1	.09	.17	**Asset 1**	1.0	-	-
Asset 2	.04	.05	**Asset 2**	0.1	1.0	-
Asset 3	-0.001	.45	**Asset 3**	0.0	0.0	1.0

Note that the maximization function is recursive. That is, the optimal allocation of wealth within a goal is influenced by the allocation of the resource pool (ϑ) to that goal, and the optimal allocation of the resource pool is influenced by the allocation of investments within a goal. This creates some implementation and computational challenges. For our test, we make use of a two-step optimization approach using a Monte Carlo engine.

The first step is to find the mix of investments which delivers maximal probability of goal achievement for each possible level of ϑ, for each goal in the goal space. This can be done for any level of granularity; we tested across whole percentage points (i.e. 1%, 2%, 3%, etc). Stated simply, for each value of ϑ and each goal, we simulate many potential portfolio allocations for consideration. We then find and return the investment allocation that offers the highest level of goal achievement probability (for each value of ϑ).

Similarly, the second step is to find the combination of goal allocations that maximizes utility. We generate many potential ϑ allocations, match the optimal investment mix with that ϑ

allocation, then return the combination with the highest utility. The subportfolio allocations are then synced with the relevant theta allocation, and the subportfolio aggregate portfolio allocations are returned.

Given the characteristics from Table 1, and following the procedure described, goal A should be given 65% of the initial resource pool, resulting in an achievement probability of 60%. goal B receives 30% of the resource pool, yielding an achievement probability of 100%. Goal C receives 5% of initial resources, which translates into a probability of achievement of 100%. This results in an aggregate portfolio of 77% to asset 1, 19% to asset 2, and 4% to asset 3.

As mentioned, optimal asset weights are dependent on the allocation of initial resources. For goal A, the preference for the lottery-like asset (Asset 3) is high when resource allocation is low, which gives way to a heavy stock-like asset (Asset 1) around $0.10 < \vartheta < 0.20$, leading to a more diversified, but heavier allocation to the bond-like asset (Asset 2) around $0.25 < \vartheta < 1.00$.

Through this example the cost of an exogenous allocation across the goal space is apparent. Suppose our agent, rather than using the method above to allocate her wealth, has exogenously allocated 2% of her wealth to goal C, 20% to goal B, and 78% to goal A. This may make some intuitive sense as goal A is clearly the most important of the three, and goal C could be categorized as a "nice to have." What does she get in exchange for pushing further wealth into goal A? Her probability of achieving A moves to 62% and the achievement of goals C and B drops to 43% and 62%, respectively. For a gain of a mere two percentage points, she probably will not achieve goal C, and goal B, which could be endowed today, is now considerably less certain. Not to mention, the role of the gamble in her portfolio (asset 3) grows substantially.

In this example, an exogenous allocation of wealth comes at the cost of two of her goals.

2. Comparison with Other Frameworks

2.1 Hidden Endogeneity of Current Across-Goal Allocation Models

It should be well noted that current models of wealth allocation across mental accounts such as Chhabra (2005); Das, et al (2010), and the extension of Alexander, et al (2017); among others; assume an exogenous allocation across the goal space. Given that the model herein is an endogenous allocation across the goal space, some discussion is warranted with respect to the former versus the latter.

In exogenous models, such as that presented by Das, et al (2010) and other authors expanding their work, it is assumed that the agent approaches the optimization problem with the allocation across goals already made. The agent then specifies an acceptable probability of failing to reach some threshold return. In a goals-based context, the threshold return is the return required to accomplish the goal, and, thus, the agent specifies the highest probability of failure she is willing to accept for a given goal. We here demonstrate that the exogenous model is, upon closer inspection, inherently endogenous.

Demonstration. Let the maximum acceptable failure probability be exogenously given as $\alpha: \Pr[r \leq H] \leq \alpha$, as are the variables of the goal space, W and t, where W is the final wealth required to vest the goal and t is the time horizon within which the goal must be accomplished. In a goals-based context, we can define $H = \left(\frac{W}{\vartheta\varpi}\right)^{\frac{1}{t}} - 1$ as the return required to achieve the goal with ϖ representing the total resource pool and ϑ representing the allocation of the resource pool to the goal. For tractability, we employ a logistic cumulative distribution and rewrite:

$$\Pr[r \leq H] \leq \alpha \Rightarrow \frac{1}{1 + \exp\left(-\frac{\frac{W}{\vartheta\varpi}^{\frac{1}{t}} - 1 - r}{s}\right)} \leq \alpha, \tag{7}$$

leading to

$$\vartheta \geq \frac{W}{\varpi\left[1 + r - s\ln\left(\frac{1}{\alpha} - 1\right)\right]^{t}}. \tag{8}$$

Therefore, an exogenous declaration of α is mathematically synonymous to a declaration of a minimum mental account allocation, ϑ. □

The Brunel (2015) model is a unique case. In the Brunel model, goals are identified and minimum thresholds of achievement are specified. Brunel's model then algorithmically determines the lowest funding cost for each goal and uses that cost to determine the value of assets needed to defease the goal given the required minimum probability of success. The wealth pool is drawn down to "pay for" the defeasement of the goals. The benefits of such an allocation solution are obvious: it incorporates all goals-based constraints (time horizon, future funding requirements, minimum probability of success, etc), it arrives at a reasonable allocation of wealth across mental accounts, and it carries an attractive simplicity. In many ways, it is an explicit recognition of the endogeneity hidden in previous work. Indeed, a simple adjustment to equation (8) reveals Brunel's insight: rather than a relationship of inequality, Brunel makes it a strict definition.[3] This simple adjustment carries quite a bit of water.

Embedded in Brunel's model, however, is the same twofold critique we level against the other, more exogenous, frameworks. First, almost never will Brunel's framework account for *all* wealth in the agent's investible pool of assets. After the allocation across goals has been made, the agent will have either an excess pool of resources or insufficient resources. In the case of excess, the model offers no insight into how to divvy the excess. In the case insufficiency, the model is infeasible (offers no solutions). In both cases, Brunel (2015) suggests the advisor return to the client and determine the best course through conversation. Second, as we have asserted, there is an implicit understanding when vocalizing a minimum probability of achievement threshold; namely, if possible, an agent would prefer a higher probability of achievement to a lower (our first axiom). Again, Brunel (2015) acknowledges this shortcoming and suggests a resolution in client conversation. Thus Brunel's model, despite its effectiveness, fails to satisfy our first axiom, instead taking an interactive[4] approach to satisfying this implicit assumption. The goals-based utility framework replaces that heuristic with a fully-formed solution, leaning on the critical insights made by Brunel (2015), Das, et al (2010), Shefrin and Statman (2000), Thaler (1985), and pushing them to their logical conclusion.

At first blush, it may seem that soliciting minimum acceptable thresholds is a more effective way to elicit goal allocations—a la Brunel's model. However, there is substantial ambiguity with respect to the overlap of allocation ranges. A trivial arrangement of equation (8) illustrates that an agent should be willing to pay a minimum of $W/\left[1+r-s\ln\left(\frac{1}{\alpha}-1\right)\right]^t$ and a maximum of W. To the extent these ranges overlap between goals, how an investor should resolve the conflict is unclear in current frameworks.[5] For strictly exogenous allocations, there is the additional problem of the potential mismatch between the amount an individual is willing to pay and the amount she actually offers to pay. In the presence of this mismatch, the exogenous model becomes self-contradictory.

Given that exogenous models are implicitly endogenous, and that the Brunel model is partially so, the goals-based framework presented in section 1.1 attempts to acknowledge and account for a fully endogenous allocation of wealth across mental accounts, keeping a consistent logic throughout. While the goals themselves and their variable vectors are exogenous, once declared, how an individual may allocate optimally between them follows as a matter of course. Indeed, any other wealth allocation across them directs capital to something other than its highest use.

Rarely in practice does an individual have the exact capital required to precisely meet their minimum threshold requirements. Sometimes an individual has more money than is necessary, but more often an individual does not have enough to meet her minimum thresholds. How does she then allocate excess capital or sacrifice one goal for another? The certainty equivalence allows for answers to these questions. With relative goal valuation we can properly understand how one goal may be substituted for another, and ultimately how surplus capital may be efficiently directed.

When compared with current frameworks of across-goal allocation present in the literature, the direct benefits of the model presented herein become more apparent. First, solutions are continuous. Infeasible results in the Brunel model and the exogenous models are feasible in the goals-based utility model. Second, the whole pool of wealth is accounted for in our model, removing any ambiguity with respect to excess wealth or insufficient wealth. Third, goals-based utility solutions, as we shall see through section 2, stochastically dominate other approaches, offering higher probabilities of goal achievement. Fourth, the goals-based utility model may explain some common normative-behavioral paradoxes present in current theories of utility, discussed in section 3.

---

[3] In other words, instead of equation (8) as presented, Brunel's model is $\vartheta = \frac{W}{\varpi\left[1+r-s\ln\left(\frac{1}{\alpha}-1\right)\right]^t}$.

[4] The interactive approach to across-goal allocation is an exogenous approach. So, while Brunel's model does endogenously allocate the *initial* pool of wealth, it is exogenous with respect to any wealth left unallocated by that initial pass. In that way, the Brunel model is a hybrid approach between strict endogenous allocation and strict exogenous allocation.

[5] For example (and with considerable loss of generality), suppose Jane has $1,500,000 to defease two goals. If Jane is willing to fund goal A with a minimum of $725,000 and a maximum of $2,000,000 and she is willing to fund goal B with a minimum of $423,000 and a maximum of $923,000, then how each framework resolves the overlap between the two (i.e. $725,000 to $923,000) is not addressed. Brunel's model simply takes the lowest amount: $725,000 for goal A and $423,000 for goal B, but is silent with respect to the remaining assets. That is, after allocating the initial $1,148,000 ($725,000 plus $423,000), Brunel's model requires the advisor to discuss with the client how to allocate the remaining $352,000 ($1,500,000 minus $1,148,000). The goals-based utility model, in contrast, removes ambiguity with respect to overlap, removes ambiguity with respect to excess, offers feasible solutions should the resource pool be insufficient to fund all goals, and offers higher probabilities of goal achievement than current frameworks.

2.2 Mean-Variance Space

As Das, et al (2010) show, the probability maximization problem is functionally the same as the mean-variance maximization problem, but only for some cases. As is also discussed by Das, et al (2010, hereafter DMSS), there are infeasible probability maximization solutions within mean-variance space. In this section, we show where the two are functionally similar, but where the goals-based procedure offers continuity across solutions which are infeasible in mean-variance (MV) space.

In the DMSS approach, the investor specifies a maximum threshold probability of the portfolio return failing to achieve the required return, $\alpha\colon \Pr[r_{port} \leq r_{req}] \leq \alpha$. This threshold can then be used to find an implied risk-aversion coefficient in mean-variance space. In effect, this approach offers a "translation" to aid an investor in the specification of their risk-tolerance level.

A quick note of context is warranted here: DMSS draw a parallel between threshold specification in the VaR sense and in the goal-based sense.[6] Whether specifying a negative number (as in the case of VaR-like metrics) or a positive one (as in the case of required return-like metrics), the math is the same. We operate under the goals-based context which assumes that individuals are better at specifying their probability of failing to achieve a goal than they are at specifying a VaR-like metric (see Brunel, 2015).

In the following, we demonstrate how the MV approach is stochastically dominated by the goals-based procedure from the perspective of goal achievement. Before discussing stochastic dominance, however, we offer a more practical critique. When asked what probability of failure one is willing to accept, is any response other than "0%" rational? Implicit in any reasonable answer is our axiomatic assertion that a lower failure probability would be preferred to the threshold specified, if possible. If that were not so, we could rightly label the agent as irrational—preferring a higher probability of failure to a lower. By acknowledging risk as the probability of failing to achieve a goal, the next logical step is to simply minimize that risk. Soliciting acceptable thresholds for risk is more about tractability in the MV framework than it is about accomplishing investor objectives. More sensible is to construct a framework to model the objective, rather than attempt to fit the objective to a pre-existent framework.

We now show that the MV approach is stochastically dominated by the probability maximization framework. Let $\sigma_1 < \sigma_2$ and $\Phi(r; \mu, \sigma)$ be the lower-tail cumulative distribution function, with an average of μ and a standard deviation of σ. Note that the lower-tail cdf is the probability of failure, so $\phi(\cdot) = 1 - \Phi(\cdot)$ in our context. The following relationships hold (these are demonstrated more formally below):

$$\begin{aligned} &1\colon \Phi(r;\ \mu, \sigma_1) < \Phi(r; \mu, \sigma_2) \quad \text{when} \quad r < \mu \\ &2\colon \Phi(r;\ \mu, \sigma_1) = \Phi(r; \mu, \sigma_2) \quad \text{when} \quad r = \mu. \\ &3\colon \Phi(r; \mu, \sigma_1\ ) > \Phi(r;\ \mu, \sigma_2) \quad \text{when} \quad r > \mu \end{aligned} \tag{9}$$

Letting the required return be $(W/\vartheta\varpi)^{1/t} - 1 = r$ and the portfolio return be $\mu\colon \sum \omega_i r_i = \mu$ we can infer that mean-variance optimization and probability maximization are synonymous so long as $r \leq \mu$, as implied by relationships (1) and (2). This was shown by DMSS. However, when $r > \mu$, mean-variance solutions become stochastically dominated by probability maximization solutions because maximal probabilities are had from *increasing* variance rather than minimizing it, as described by relationship (3).

Because real-world investors are bounded in their ability to borrow and sell short, there must exist some endpoint of the mean-variance frontier, (σ_e, μ_e). Further, let $\Phi(r;\ \mu_e, \sigma_e)$ be the goal-implied failure probability of this endpoint. When $r > \mu_e$, MV solutions will maintain exposure to the endpoint portfolio (or be declared infeasible), while probability maximization solutions will seek higher variances: $\sigma_2 > \sigma_e \Rightarrow \Phi(r; \mu_e, \sigma_e) > \Phi(r; \mu_e, \sigma_2)$. Relationship (3) shows that MV solutions will be stochastically dominated so long as higher variance solutions exist, even though these solutions may carry lower expected returns (though there is a trade-off between return and variance). Figure 1 illustrates this stochastic dominance.

That MV optimization is stochastically dominated by probability maximization should come as little surprise. Markowitz (2010) discusses the decision to abandon a probability maximization form of utility and adopt a quadratic form. In short, the quadratic form of mean and variance is a decent approximation of probability ranking. It is, however, an approximation, and, as we have subsequently learned, is not feasible in all cases given real-world constraints.

Alexander, et al (2017) present an insightful extension of the DMSS model. Primarily, their work extends DMSS in light of estimation risk. Though we do not explore it here, optimization in the presence of estimation risk is recommended. More salient to this discussion, however, is their extension that offers variable thresholds which ensure MV optimal portfolios will exist. While promising, their work does

[6] Table 2 of their paper illustrates this, but it is more explicitly stated at the top of page 314.

not counter our critique because we propose that goal-based investors are limited in their ability to borrow and sell short (and thus there exists some endpoint on the MV frontier). Even under their variable threshold paradigm, short-selling and leverage must be allowed to guarantee MV optimal portfolios exist, both of which are commonly disallowed in mental account portfolios.

In a similar vein, others in the literature simply constrain-away this critique. For example, Elton, et al (2009) impose the constraint that the threshold return must be less than the mean return of a portfolio. This constraint, however, fails to properly acknowledge how investors actually operate. Shefrin and Statman (2000), Statman (2004), Chhabra (2005), and Brunel (2015) all acknowledge the existence of "aspirational" goals. What are aspirational goals if not return requirements above those on offer by a traditional investment space?

The removal of such constraints is important because it restricts the individual to a budget for aspirational goals. Under our model it may, at times, be rational to gamble, but certainly not to gamble everything (gambling is also addressed in section 3.2). In addition, the removal of this constraint may solve some common puzzles of utility—that is, it explains some observed data (see section 3). More than this, however, a goals-based model offers continuity—accounting for and answering basic questions about goals that traditional finance simply constrains away.

For investors who wish to rationally allocate across goals, but who also desire to remain efficient in MV space, the following adaptation applies. Generate an efficient frontier subject to an investor's investment universe and borrowing/short-sale constraints. Let (σ_e, μ_e) be the endpoint of the frontier such that μ_e and σ_e are the highest possible values of the frontier. Any point on the frontier, (σ, μ), may be converted to a probability of outcome given a goal vector: $\exists A \coloneqq (\varpi\vartheta, W, t) \wedge (\sigma, \mu) \Rightarrow \phi\left[\left(\frac{W}{\varpi\vartheta}\right)^{\frac{1}{t}} - 1; \mu, \sigma\right]$. As discussed (and shown more formally below), when $\left(\frac{W}{\varpi\vartheta}\right)^{\frac{1}{t}} - 1 \leq \mu_e$, MV efficiency is equivalent to probability maximization, and when $\left(\frac{W}{\varpi\vartheta}\right)^{\frac{1}{t}} - 1 > \mu_e$ probability maximization solutions are inefficient in MV space. The maximal utility function, then, becomes:

$$\max_{\vartheta,\omega} \sum_{i}^{N} \begin{cases} v(i)\phi\left[\left(\frac{W}{\varpi\vartheta_i}\right)^{\frac{1}{t_i}} - 1; \mu, \sigma\right] & \forall \quad \left(\frac{W}{\varpi\vartheta_i}\right)^{\frac{1}{t_i}} - 1 \leq \mu_e \\ v(i)\phi\left[\left(\frac{W}{\varpi\vartheta_i}\right)^{\frac{1}{t_i}} - 1; \mu_e, \sigma_e\right] & \forall \quad \left(\frac{W}{\varpi\vartheta_i}\right)^{\frac{1}{t_i}} - 1 > \mu_e \end{cases} . \tag{10}$$

In other words, for return requirements that exceed the maximal return offered by the MV frontier (σ_e, μ_e), exposure to the endpoint of the frontier is maintained. For all other cases, the optimization procedure is the same as either MV or probability maximization. Though a heuristic (and stochastically dominated by strict probability maximization), the structure illustrated in equation (10) has the advantage of eliminating the infeasible solutions present in the DMSS framework, and it optimally allocates resources across goals subject to MV constraints.

Demonstration. We here demonstrate that probability maximization is synonymous with variance minimization when $r < \mu$, relationship (1). Let the probability of failure be defined by the logistic cumulative distribution function, $\Phi(r;\, \mu, s) = \frac{1}{1+\exp\left(-\frac{r-\mu}{\sigma}\right)}$, and we are concerned with the truthiness of $\Phi(r; \mu, \sigma_1) < \Phi(r; \mu, \sigma_2)$. Further, let $r - \mu = a \therefore r < \mu \Rightarrow a < 0$, and recall that $\sigma_1 < \sigma_2$:

$$\frac{1}{1 + \exp\left(-\frac{a}{\sigma_1}\right)} < \frac{1}{1 + \exp\left(-\frac{a}{\sigma_2}\right)}, \tag{11}$$

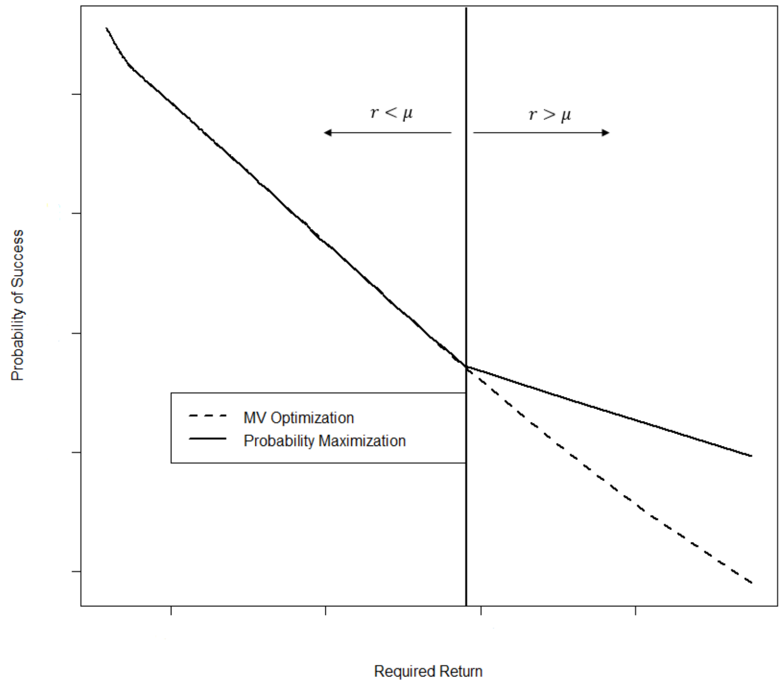
leading to $-\frac{a}{\sigma_2} < -\frac{a}{\sigma_1}$, which yields $\sigma_1 < \sigma_2$, reflective of relationship (1). Therefore, it must be so that the probability of failure is minimized when variance is minimized, so long as $r < \mu$.

We now demonstrate that probability maximization is synonymous with variance maximization when $r > \mu$. Again, let $a = r - \mu \therefore r < \mu \Rightarrow a > 0$, and we set up the logistic cdf. We are concerned with when $\Phi(r; \mu, \sigma_1) > \Phi(r; \mu, \sigma_2)$ is true:

$$\frac{1}{1 + \exp\left(-\frac{a}{\sigma_1}\right)} > \frac{1}{1 + \exp\left(-\frac{a}{\sigma_2}\right)} \tag{12}$$

leading to $-\frac{a}{\sigma_2} > -\frac{a}{\sigma_1}$, which yields $\sigma_1 < \sigma_2$, consistent with relationship (3). Therefore, it must be so that the probability of failure is minimized when variance is maximized, so long as $r > \mu$. In this way, mean-variance optimization is a probability maximization special case. □

FIGURE 1 Probability Maximization Stochastically Dominates Mean-Variance Optimization



Example. Using the asset characteristics described in Table 1, it is a simple matter to trace the MV efficient frontier.[7] Coupled with the characteristics of the individual presented in section 1.2, it can be shown that the goals-based optimization procedure results in a portfolio which is efficient in MV space. Panel A of Figure 2 illustrates this result, as the optimal goals-based portfolio resides on the efficient frontier.

However, as the total pool of initial resources shrinks beyond a certain limit (or the time horizon is sufficiently shortened), solutions become either infeasible in MV space or inefficient in the goals-space. In those moments, the optimal goals-based portfolio becomes MV inefficient, as illustrated by Panel B of Figure 2. In this scenario, the initial pool of resources was reduced from \$972,000 to \$150,000. Because of the lowered levels of resources (and thus the higher return requirements), this move away from the MV frontier is driven by an increased reliance on the high-variance, lottery-like asset (Asset 3).

Figure 3 illustrates where MV solutions become inefficient (or infeasible) from a goal-achievement perspective in the case study discussed. As the share of initial resources allocated to a goal decreases, the goals-based optimization procedure begins to stochastically dominate MV portfolios from the perspective of goal-achievement probability.

In MV space, if all mental-account sub-portfolios reside on the efficient frontier and short-selling is allowed, then the aggregate portfolio will also reside on the efficient frontier (see Brunel 2006). As we have shown, however, not all sub-portfolios will necessarily reside on the frontier. In cases where a goal carries a low value ratio, the allocation granted to the goal may be sufficiently low to warrant a departure from the MV efficient frontier. Alternatively, it is possible that the initial pool of wealth is sufficiently low to warrant justified exposure to portfolios which lie off of the MV efficient frontier. Such subportfolios will not be MV efficient (and thus the aggregate portfolio will not be MV efficient), though they will provide the maximum probability of goal achievement.

As we have shown, for investors concerned with MV efficiency, the goals-based optimization procedure may be suboptimal. For investors who do wish to remain MV efficient,

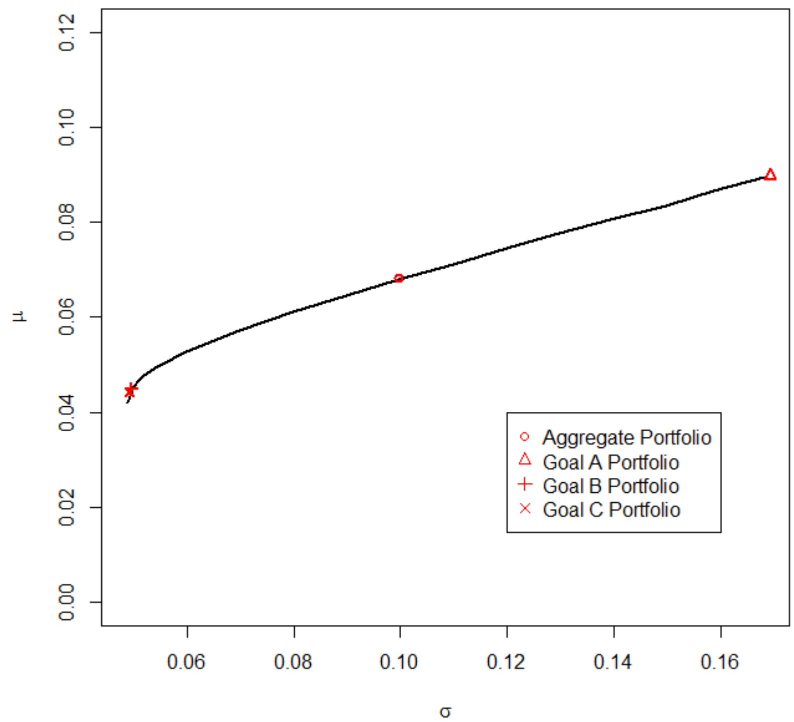
[7] Because they are common constraints on goal-based investors, we have assumed no short-sale and no-leverage constraints: $\sum_i \omega_i = 1, \omega \geq 0$ where ω is the investment weight.

equation (10) offers a structure that eliminates infeasible solutions while maintaining the rational allocation across goals. Alternatively, the procedure here may also be modified to check whether the required portfolio return is larger than the maximal return offered by the investible MV universe. When this is so, the solution is infeasible, and some modification is required to the goal vector—either the time horizon increased, or the final required wealth must be reduced. For investors concerned with maximizing the probability of achieving specified goals, however, we have shown that the goals-based procedure stochastically dominates the MV approach.

But whether investors choose to remain MV efficient is not the main thrust of the broader discussion. We are less concerned with the allocation across investments and more concerned with the allocation across goals. Because many investors are MV constrained, in this section we have attempted to demonstrate two alternatives available to portfolio managers: strict probability maximization, and probability maximization subject to MV efficiency.

FIGURE 2

PANEL A: Goals-Based Portfolios Which Reside on the Mean-Variance Efficient Frontier



PANEL B: Across Infeasible Mean-Variance Solutions, Goals-Based Solutions Depart from Efficient Frontier

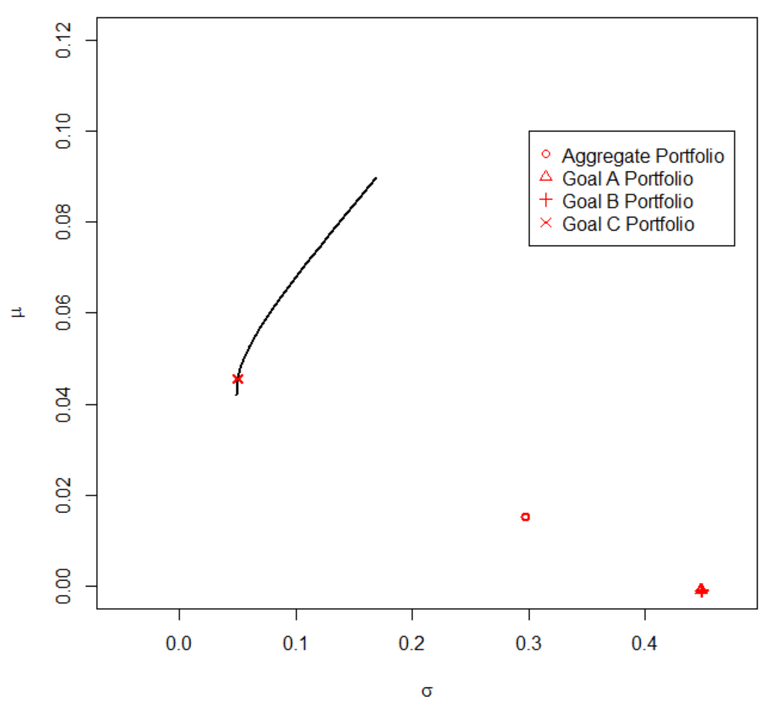
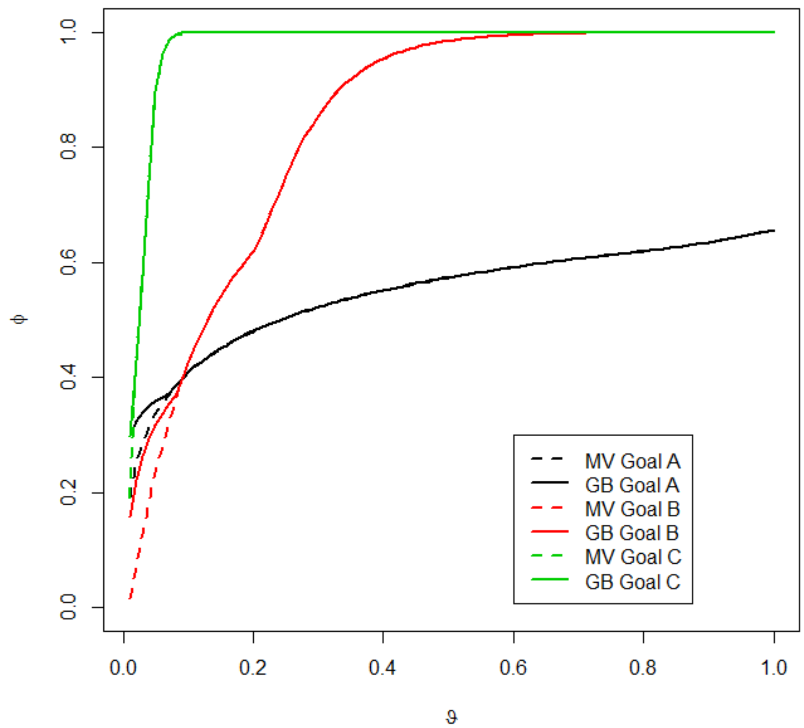


FIGURE 3 Goal-Achievement Probability as a Function of Initial Wealth Allocation, Mean-Variance vs. Goals-Based Optimization



2.3 Comparison with Behavioral Portfolio Theory

BPT describes portfolios on a "behavioral-wants" frontier, making a trade-off between accomplishment probability and expected return. Other emotional and expressive benefits can be incorporated, according to Brunel (2015), whereas such non-quantitative portfolio aspects are eliminated from a MV approach. Of importance to this work is the "layered" and mental accounting structure afforded by the BPT framework. Most seminal, however, is BPT's redefinition of risk as goal failure.

Our work both draws from and validates many conclusions from this theory. Assuming goal achievement across multiple, mutually-exclusive, and differently-valued goals is paramount for an economic agent, we have shown how an economic agent may rationally divide assets into such layers as described by BPT. This division will often yield allocations to non-MV-efficient portfolios, again, as described by BPT. Agents will be variance-seeking when aspiration levels are high enough, as described by BPT. Also similar is that BPT agents assign unique values to each of their "layers," resulting in different wealth allocations for each.

Other than showing that an agent will allocate to safety layers first, BPT does not offer a specific utility function with regard to how assets should be rationally divided across the goal space, a point of difference. Further, BPT agents seek to maximize both probability of achievement *and* expected wealth, while probability maximizers are concerned only with maximizing probability of goal achievement. This is a subtle yet important difference. As a result of this trade-off, BPT necessarily yields stochastically dominated portfolios. By definition, such a utility function must create the possibility to choose a portfolio which offers higher potential wealth, but with higher probability of failure. This is a point discussed by Shefrin and Statman (2000) as the BPT efficient frontier explicitly illustrates this tradeoff between the probability of failure and expected wealth. This fact creates a violation of our first axiom: that one prefer a higher probability of achievement to a lower. By contrast, probability maximizers select portfolios which offer the highest probability of goal achievement *only*, and will therefore stochastically dominate portfolios which balance the two. There is no frontier—efficient or otherwise—for probability maximizers. That said, the theory presented here may be thought of as a BPT special case where the weighting of expected wealth and probability of failure is zero and one, respectively.

This may be readily demonstrated through observation of the Cobb-Douglas-like BPT utility function: $u_{\text{BPT}} = [1 - \Phi_{\text{BPT}}(r; m, s)]^{1-\gamma}[(1+m)^t]^\gamma$, where $0 < \gamma \leq 1$.[8] When $\gamma = 0$, no weight is given to portfolio return (other than through the cdf), and utility reduces to probability maximization. However, when $\gamma > 0$, there is some tradeoff made between the two—that is, the investor is willing to forego some achievement probability in exchange for higher return. In that case, BPT utility maximizers will be stochastically dominated by strict probability maximizers. It is worth noting that probability maximization is a BPT special case, and because MV maximization is a probability maximization special case, it is a BPT special case as well.

For investors who wish to remain BPT-efficient within goals, but would like to rationally allocate across goals, the utility maximization function becomes the following:

$$\max_{\vartheta,\omega} \sum_i^N v(i)\phi\left[\left(\frac{W}{\varpi\vartheta_i}\right)^{\frac{1}{t_i}} - 1; m, s\right]^{1-\gamma} [(1+m)^{t_i}]^\gamma. \quad (13)$$

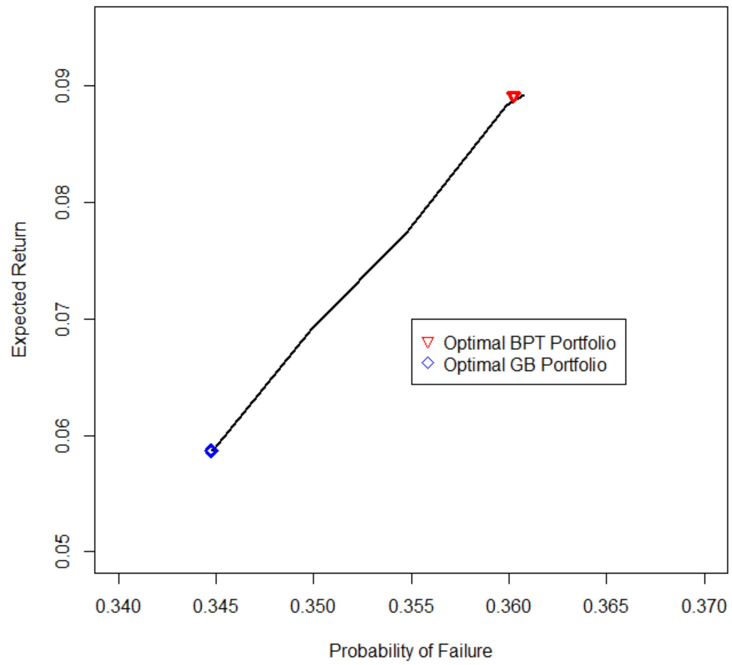
Example. Let BPT utility be a Cobb-Douglas-like tradeoff between failure probability and expected wealth: $u = [1 - \Phi(x; m, s)]^{1-\gamma}\,[(1+m)^t]^\gamma$. Continuing with the asset characteristics presented in Table 1 and the agent details presented in the example, let us examine the one-goal case of goal A with $\gamma = 0.04$.

The optimal BPT portfolio solutions yield an efficient frontier—a tradeoff between expected return and probability of failure. As gamma increases, the optimal BPT portfolio moves up the frontier. Because the goals-based optimization procedure makes no tradeoff between expected return and risk, it is always on the frontier, but always at the lower endpoint and is therefore stochastically dominant. In the results illustrated by Figure 4, the goals-based portfolio stochastically dominates the BPT portfolio: a 34.5% versus a 36.0% probability of failure.

As with our discussion on probability maximization under MV efficiency, it is not our aim to argue for or against BPT efficiency. We are primarily concerned with the allocation of wealth across goals. The theory of portfolio allocation across assets is well-researched and well-understood. Even so, in this section we have attempted to demonstrate how portfolio managers may implement the meta-theory of goal allocation in two paradigms: strict probability maximization, and probability maximization with some allowance for higher expected return (i.e. BPT efficiency).

[8] It should be noted that γ must be constrained in this manner even though Shefrin and Statman (2000) do not explicitly say so. If $\gamma > 1$, their formulation yields increasing utility for *increasing* failure probability—a paradoxical result.

FIGURE 4 The BPT Efficient Frontier



3. Some Common Puzzles of Utility

In an effort to provide some empirical support for this approach, we apply goals-based utility to some common utility puzzles. In this section, we analyze the Samuelson Paradox, the Friedman-Savage Puzzle, and probability weighting functions.

3.1 The Samuelson Paradox

In his 1963 paper, Samuelson offered a colleague a bet: one coin flip, heads pays $200 and tails owes $100. The colleague refused but countered that he would be willing to accept 100 such flips rather than just one. Samuelson goes on to explain how this is an irrational response—an individual who is unwilling to lose $100 should be unwilling to lose $100 one hundred times, even though that may be an unlikely possibility. Rabin (2000) and Rabin and Thaler (2001) use this as a launching point to declare expected utility theory a "dead theory." We posit that Samuelson's colleague may have behaved rationally in the context of his goal-achievement desire.

We understand the colleague's status-quo utility as $u(\text{baseline}) = v(A)\phi(w)$,[9] and the utility of Samuelson's offer is

$$u(\text{offer}) = v(A)[0.5\phi(w + w^+) + 0.5\phi(w - w^-)] \quad (14)$$

We are concerned with whether $u(\text{offer}) < u(\text{baseline})$ is true,

$$v(A)[0.5\phi(w + w^+) + 0.5\phi(w - w^-)] < v(A)\phi(w). \quad (15)$$

Simplifying to an interesting result:

$$\frac{\phi(w + w^+) + \phi(w - w^-)}{2} < \phi(w). \quad (16)$$

In other words, if the average of the resultant upside probability and downside probability is less than the baseline, the colleague should prefer the baseline.

In what cases could this be so? If the wealth function is monotonic and non-linear, as it is in the cumulative probability function, this constraint is met. Specifically, when the probability gain offered by Samuelson's $200 is a smaller positive probability difference than the loss of $100. Equation (16) may be rewritten to illustrate this identity: $\phi(w + w^+) - \phi(w) < \phi(w) - \phi(w - w^-)$. Figure 5 illustrates this asymmetric probability payoff. In Figure 5, the resultant probability of goal-achievement is the cumulative distribution

[9] Our standard form of goal achievement probability is a function of three variables: $\phi(w, W, t)$. However, in an effort to increase readability, we have reduced the form to one variable as the other two are constant.

function where higher required returns (because of lower levels of initial wealth) make goal-achievement less likely.

Assuming 100 coin flips is a negligible increase in the time investment, the colleague's counter-offer is a different game altogether. Because of the nature of probabilities, we can see that the probability of gain is near certain. We learn from Rabin and Thaler (2001) that the probability of *any* loss is a mere $1/2300$. Let $\Pr[w^-] = \frac{1}{2300}$ and $\Pr[w^+] = 1 - \frac{1}{2300} = \frac{2299}{2300}$ and let us assume that *any* loss is to be avoided (which is an excessive assumption). We are concerned with the truth of the inequality $u(\text{counteroffer}) > u(\text{baseline})$:

$$v(A)\left[\frac{1}{2300}\phi(w - w^-) + \frac{2299}{2300}\phi(w + w^+)\right] > v(A)\phi(w), \quad (17)$$

which simplifies to

$$\frac{\phi(w - w^-) + 2299\phi(w + w^+)}{2300} > \phi(w). \quad (18)$$

It would be a very steep function of wealth to yield a denial of this offer no matter how extreme the loss, as the resultant probability of a win (w^+) is weighted 2299 times more than the resultant probability of a loss (w^-). Taken another way, we know that the lower-bound of probability is zero: $0 \leq \phi(w - w^-) \leq \phi(w)$. In a worst-case scenario where the loss presented is catastrophic, i.e. $\phi(w - w^-) = 0$, the problem simplifies to:

$$0.999565\phi(w + w^+) > \phi(w). \quad (19)$$

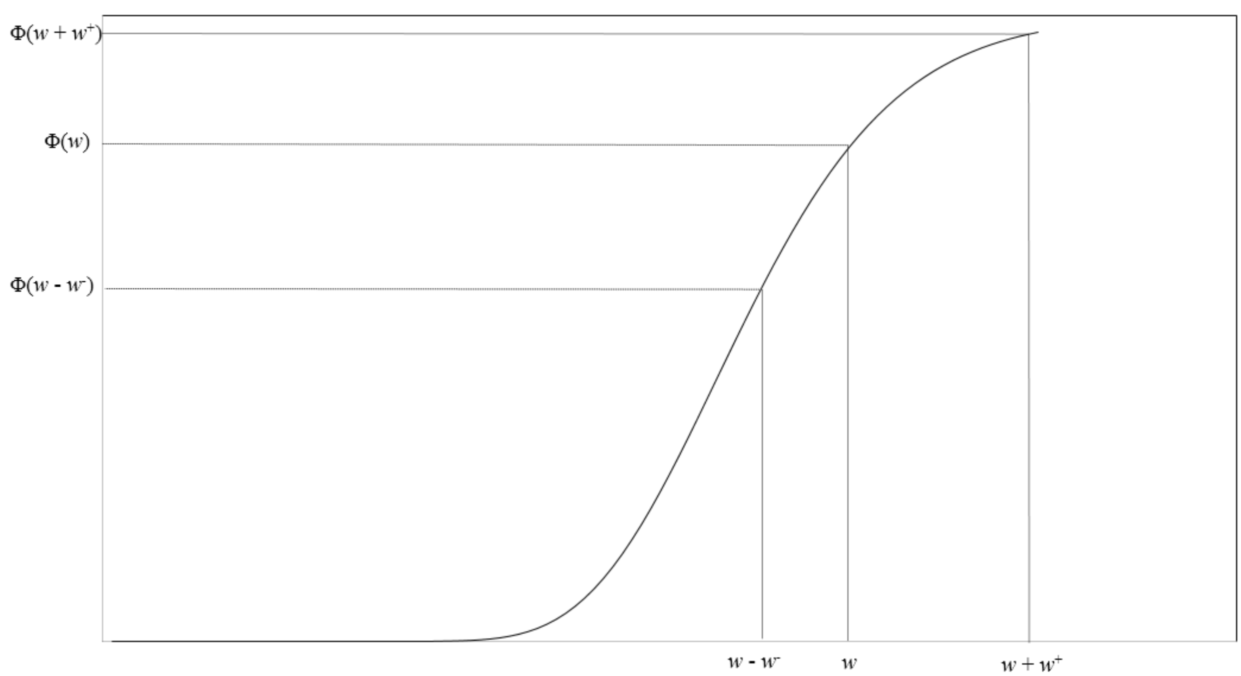
In other words, so long as the new probability is 1.004369 times the old, the colleague's offer should be accepted (a move, for example, from 62.000% to 62.271%). This validates Rabin and Thaler's (2001) intuitive conclusion that "A good lawyer could have you declared legally insane for turning down this gamble." More intuitively, we can understand that the $\phi(\cdot)$ function moves from concave to convex. A status quo (i.e. current wealth) around this inflection point means that a loss will be felt more significantly than a gain of equal value.

For Samuelson's offer, it would appear that one could expect acceptance under some circumstances, but in real-world scenarios it may also be rejected. We do not know the details of his colleague's utility function. However, under almost no real-world scenarios would we expect a goals-based probability maximizer to reject his colleague's counteroffer.

3.2 The Friedman-Savage Puzzle

The Friedman-Savage puzzle, proposed by Friedman and Savage (1948), asks why people would simultaneously buy insurance and lottery tickets. Their original solution was that the utility of wealth is a continuous curve with first and second derivatives, and that the second derivative alternated between positive and negative values across varying levels of wealth. They proposed that there must exist at least two points on this

FIGURE 5 Probability Maximization Utility Curve



curve where the second derivative equals zero. At those points—the transition from concave to convex and from convex to concave—the agent is willing to buy both insurance and lottery tickets. This is known as the double inflection solution. Markowitz (1952b), among others, offered a critique of this solution.

The framework on offer here posits that an agent allocates initial wealth across multiple goals. For goals which carry high probability of achievement, the utility of wealth function is concave and variance-aversion is present. For those goals which carry low probability of achievement, the utility of wealth function is convex and variance-affinity is present. Rather than a double inflection on a single utility curve, mental accounting proposes multiple utility curves. Agents, therefore, can be both variance-seeking and variance-averse simultaneously—buying both insurance and lottery tickets.

In addition to the multiple utility curves of mental accounting, the goals-based framework resurrects the so-called single inflection solution offered by Markowitz (1952b). Markowitz's solution looks remarkably like a cumulative probability function (see Figure 4 of Markowitz 1952b, p.154). Coupled with his assumption that the utility function is bounded both from above and below and we have most of the solution on offer from goals-based utility.

Demonstration. Let $\{A\} \in G$, and $u(G) = v(A)\phi(w_A, W_A, t_A)$, or the expanded Gaussian form

$$u(G) = v(A)\left[1 - \int_{-\infty}^{r} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(r-\mu)^2}{2\sigma^2}\right)\right] \quad (20)$$

where $r = \left(\frac{W}{w}\right)^{\frac{1}{t}} - 1$. Risk affinity is exhibited when $u'' > 0$ and risk aversion is exhibited when $u'' < 0$. Taking the second derivative of u with respect to r yields

$$u''(G) = -v(A)\frac{\mu - r}{\sigma^2}\sqrt{\frac{1}{2\pi\sigma^2}}\exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right). \quad (21)$$

Because $\exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right) > 0$ and $\sqrt{\frac{1}{2\pi\sigma^2}} > 0$, it must be so that when $v(A) > 0, u''(G) > 0\ \forall r > \mu$ and $u''(G) < 0\ \forall r < \mu$. Thus when the required return, r, is greater than the expected return, μ, an agent is risk affine. Conversely, when the required return is less than the expected return, an agent is risk averse.

In what situations would $r > \mu$ or $r < \mu$? Recall our definition $w = \vartheta\varpi \Rightarrow r = \left(\frac{W}{\vartheta\varpi}\right)^{\frac{1}{t}} - 1$, where ϖ is the total pool of initial resources available and ϑ is the allocation of those resources given to a particular goal. Since $\mu, W, \varpi,$ and t are fixed, $r > \mu \Rightarrow \left(\frac{W}{\vartheta\varpi}\right)^{1/t} - 1 > \mu$, solving for ϑ yields

$$\begin{gathered} r > \mu \text{ when } \vartheta < \frac{W}{\varpi(\mu+1)^t} \\ r < \mu \text{ when } \vartheta > \frac{W}{\varpi(\mu+1)^t} \end{gathered}. \quad (22)$$

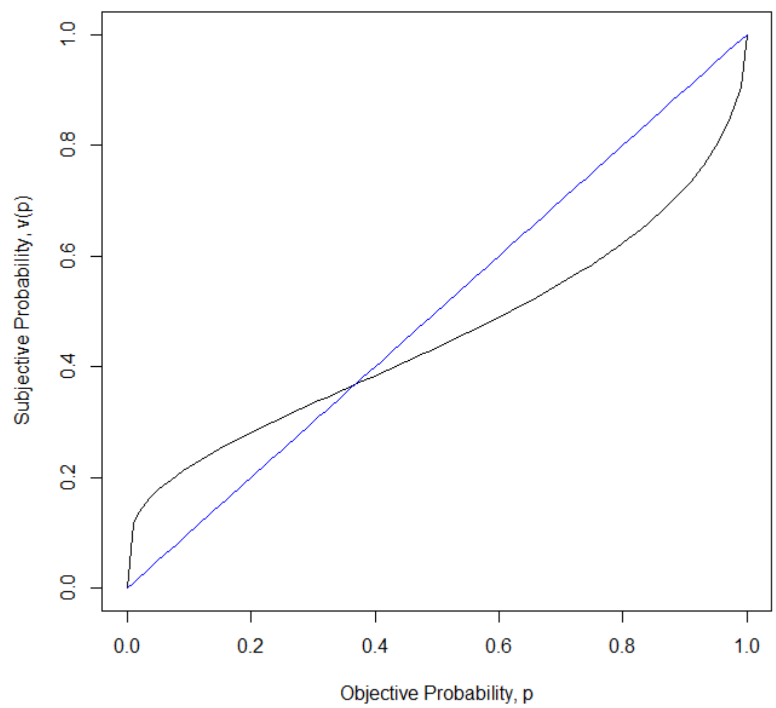
An echo of the result in section 2.2. □

3.3 Probability Weighting Functions

An important component in the development for descriptive theories of expected utility is the apparent observation that people transform objective cumulative probability weights into subjective ones. Quiggin (1982) was among the first to quantify this behavior in his model, Rank-Dependent Utility (RDU). RDU subsequently became an important component in Tversky and Kahneman's (1992) Cumulative Prospect Theory. Prelec (1998), among others, have further refined the probability weighting function. The evidence appears to show that individuals do not weight probabilities objectively, instead following an *s*-shaped transformational function, with probabilities below ~30% apparently overweighted and probabilities above ~30% apparently underweighted (see Figure 6). Prelec (1998) models this function as

$$v(p) = \exp(-\beta(-\ln p)^{\alpha}). \quad (23)$$

FIGURE 6 Prelec's Probability Weighting Function



In testing for these subjective probability weights, it is standard practice to determine the "certainty equivalent" of an offer. The certainty equivalent is the amount of money an individual would be willing to receive with certainty which makes him indifferent to an offered gamble. The certainty equivalent divided by the prospect is then determined as the subjectively-weighted probability. A subject is offered the choice between a gamble, $(w_p, p; w_{1-p}, 1-p)$, or a sure thing, $(w_s, 1)$ where w_s is varied.[10] The point at which the subject is indifferent to the gamble over the sure thing is dubbed the certainty equivalent of the gamble, and the subject's weighted probability, $v(p)$, is assessed as the certainty equivalent divided by the prospect, $v(p) = w_s/w_p$. For a full description of this experimental procedure, we refer the reader to Gonzalez and Wu (1999), and Tversky and Kahneman (1992).

We posit that these experimental results are also explicable through the goals-based utility function presented herein. We can understand the certainty equivalent as the point where the utility of the gamble equals the utility of the sure thing, $u(\Pr[w_p] = p) = u(\Pr[w_s] = 1)$, or

$$p \cdot \phi(w + w_p) + (1-p) \cdot \phi(w + w_{1-p}) = \phi(w + w_s). \quad (24)$$

Note that w is the initial wealth dedicated to the given goal, we expect $\phi(\cdot)$ to be Gaussian, and that agents will evaluate wealth offers in the context of goal-achievement, of which wealth is an input.

In our tests, we varied the values of w, m, and s, but in all cases we assumed $W = w_p$. That is, we expect the subject's goal to be accomplished by the highest amount of wealth on offer. The test involved algorithmically finding the point where the certainty equivalent equation was true by varying w_s from $p = 0$ to $p = 1$. In all cases, we found the resultant w_s/w_p to be in-line with the observed Prelec-style probability weighting function. Figure 7 illustrates our results. As with the standard Prelec Function, this result is *s*-shaped, convex at about $p < 0.35$ and concave at about $p > 0.35$. Further, the point of

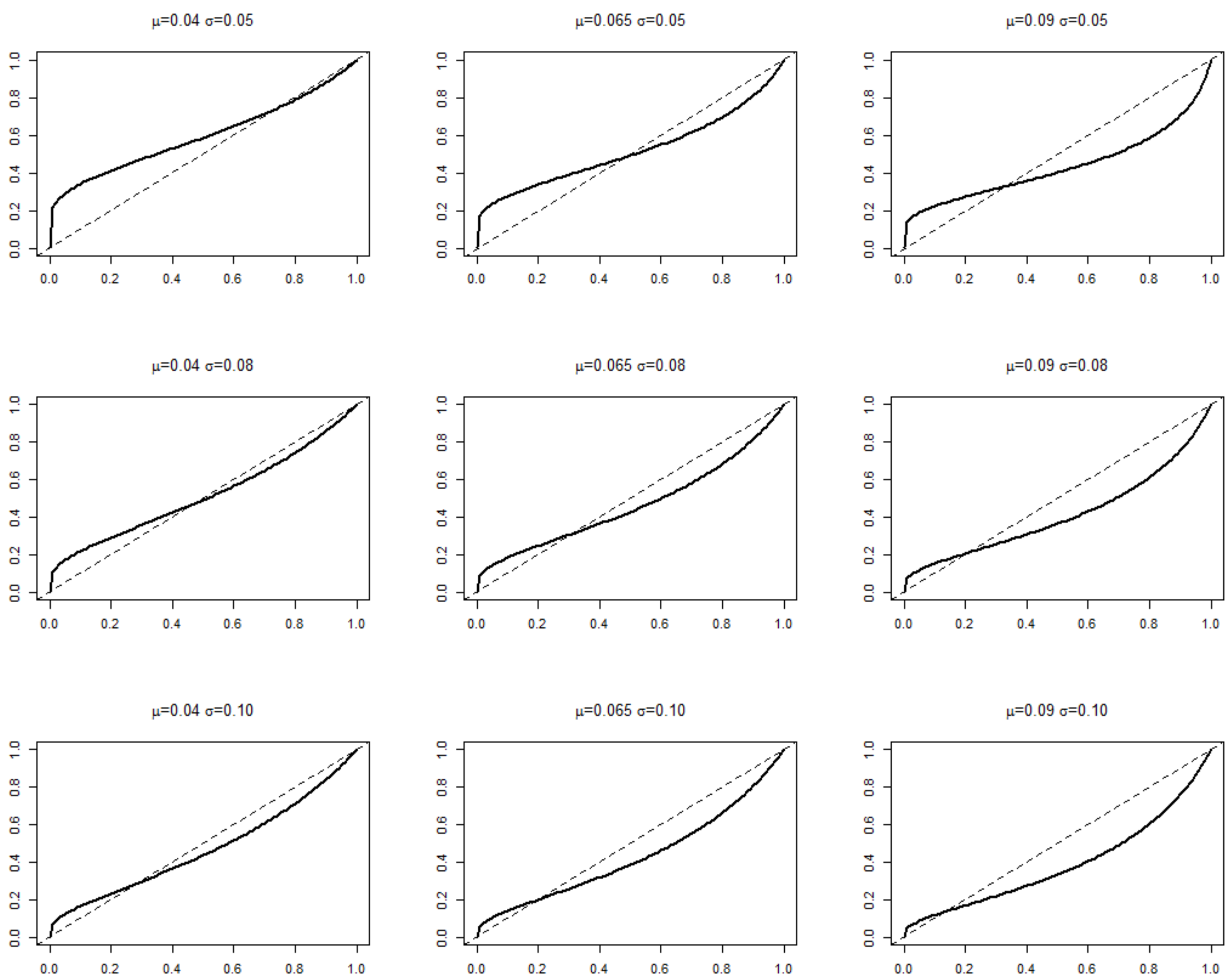
[10] Typically $w_{1-p} = 0$

transition from convex to concave, and the point of transition from "overweight" to "underweight" is in the 0.30 to 0.40 range, but malleable based on the subject's *s, m,* and *w*.

This result shows that, rather than transforming probability weights, subjects may be evaluating prospects (and thus certainty equivalent) in the context of a probability function of wealth. Further, this may explain the wide range of results catalogued by Tversky and Kaneman (1979), Tversky and Kahneman (1992), and Gonzalez and Wu (1999)—each subject would have a unique goal space and vector of variables, as well as unique average and standard deviation expectations, but they would all process their wealth preferences through some goal-achievement probability function.

FIGURE 7 Resultant Certainty Equivalents from Goals-Based Utility

y-axis is the raio of the certain offer to the lottery offer, w_s/w_p*. Tested across various levels of expected return and variance.*



4. Some Discussion

There are two layers to a goals-based theory of utility. There is the utility of the goals themselves, how they are valued and how they interact. Then there is the probability-maximization component: that is, the attempt to maximize the probability of achieving goals within the goal-space. We argue that an economic agent should allocate resources across the goal-space as a function of the relative value of one goal to another. In addition, as each goal carries its own vector of variables, the agent must balance the optimal allocation within each goal. This is not straightforward as they are mutually dependent, and thus recursive.

When compared with frameworks present in the literature, the direct benefits of the model presented herein become clear. First, solutions derived from goals-based utility are continuous.

Infeasible results in the Brunel model and the exogenous models are feasible in the goals-based utility model. Second, goals-based utility removes ambiguity with respect to the allocation of excess wealth or insufficient wealth (the majority of real-world situations). Client conversations will, of course, augment any result. Third, goals-based utility solutions offer higher probabilities of goal achievement due to their stochastic dominance. Fourth, the goals-based utility model may explain some common normative-behavioral paradoxes present in current theories of utility.

While the allocation across goals is uncontroversial, the probability maximization scheme within goals is a firm break with traditional normative theories of utility. It carries several implications, none of which should be accepted without diligent thought. That the utility of wealth is not everywhere convex represents a firm break with expected utility theory (EUT). At first blush, this appears to be problematic. We argue, however, that this is a fundamental feature of the framework.

A consequence of nonconvex utility is that an agent will accept a gamble rather than a certain outcome, which can be shown to yield a negative risk premium. How is this rational? Recall our opening axiom: an agent prefers a higher probability of goal achievement to a lower. As we have shown, there are moments where a more certain outcome yields a lower probability of goal achievement, and it would therefore violate our axiom to accept it over a less certain, but higher probability, outcome. Certainty, in and of itself, is not the objective. More intuitively and in the extreme case: given the choice between certain failure and possible achievement, one would be irrational to choose certain failure.

By way of illustration: imagine an unfortunate individual, having borrowed $10,000 from a mafia loan shark, has $7,000 and full repayment due in the morning. As anything less than repayment carries the same result—a hospital visit—the individual wanders into a casino in an attempt to gain the required $3,000 by morning. On his way in, a religious missionary stops him with an offer: if he agrees not to gamble, the missionary will give him $1,000 right now.

Such an extreme example illustrates the failing of EUT. EUT would advise our individual to prefer the sure gain to the uncertain one, even though such advice is obviously foolish in this example. In this instance, EUT has failed to account for "normal people," and not even in the sense implied by Statman (2018), where "normal people" unintentionally commit cognitive errors. It is clearly not through cognitive error that our unfortunate individual turns down the sure thing. Of course, EUT may predict the preference of a gamble over the sure thing, but once proven risk-affine we should expect him to gamble until the $7,000 was gone. There is no point of satiation.

By contrast, when viewed through the lens of goals-based probability maximization, the decision of the man to forego the missionary's offer make sense: he will use whatever resources available to maximize his chances of avoiding the hospital in the morning. This can only be accomplished through a high-variance outcome, such as gambling. He is not variance-seeking for its own sake, he is variance-seeking with an end in mind. What's more, this example illustrates his satiability for wealth; once he attains the balance of the loan, he will cease to risk current wealth in pursuit of further wealth.

We are not so bold as to assert that individuals are always rational. Bounded rationality is a sensible theory, and satisficing models continue to yield fruit. Indeed, models such as those presented by Navarro-Martinez, et al. (2018) may have implications here, as it seems reasonable to expect that agents rarely have complete and objective data with which to make decisions.

Ultimately, however, we are concerned with a normative theory rather than a descriptive one. In this discussion we simply assert that economic agents should seek to maximize the probability of achieving their goals. Once that basic premise is accepted, the whole of the goals-based probability maximization framework follows—a framework which is not new. Mental accounting (vaguely) and BPT (explicitly) acknowledge the goal-failure-as-risk framework. It is the small innovation of this discussion to fill a hole in the conversation.

If, then, one is willing to accept this initial premise, strictly pursuing mean-variance optimization is suboptimal, as goals-based probability maximization stochastically dominates mean-variance optimization. We have shown how the approach herein may be adapted to remain efficient in the mean-variance space, but such an adaptation will carry infeasible solutions, and those infeasible solutions must be accounted for using heuristics. BPT offers the closest approximation to probability maximization, but it carries an explicit tradeoff between excess return and probability of failure. This raises the simple question: why make the tradeoff at all? If the goal space is complete (as we assume in our axioms), to what is any excess wealth dedicated? Implicit is the assumption that resources are an end in themselves. We propose that resources are not the ends, they are only the means by which goals are accomplished. Absent some goal, then, there is no utility in excess resources. Not to mention, such a tradeoff violates our opening axiom, that one prefer a higher probability of goal achievement to a lower.

But, more than this, neither mean-variance theory nor behavioral portfolio theory offer a solution to how resources should be divvied up *between* goals. While we certainly argue for a probability maximization scheme, the main thrust of this discussion is less about how investments are allocated within goals as this is a well-researched and well-understood area of portfolio theory. It is the central aim of this discussion to show how one might rationally allocate resources *across* goals—how to substitute one goal for another. In this discussion, we have attempted to demonstrate how such a "meta-theory" might be implemented in its pure form, within the confines of mean-variance efficiency, or subject to an expected return tradeoff such as behavioral portfolio theory.

As individuals and institutions regularly face mutually-exclusive goals over varying time horizons, this is not some abstract question. As it comes time to vest short-dated goals, we can now better understand whether vesting comes with an acceptable loss of achievement probability for the longer-dated goals. And that understanding can now be rational and quantitative, rather than heuristic and conversational. Such theory is also practical, especially as asset management becomes increasingly automated. Automated solutions do not have the luxury of conversation to work through goal priorities and acceptable tradeoffs. Even with the benefit of conversation, however, human practitioners currently have only a heuristic approach to answering how an agent may rationally substitute one goal for another. When asked whether a goal can be vested, clients expect a right answer. Only with a cohesive approach, firmly grounded in rationality, can one be reliably given.

REFERENCES

Alexander, G.J., A.M. Baptista, S. Yan. (2017) "Portfolio Selection with Mental Accounts and Estimation Risk" *Journal of Empirical Finance,* Vol 41, pp. 161-186.

Baptista, A.M. (2012). "Portfolio Selection with Mental Accounts and Background Risk" *Journal of Banking and Finance*, Vol 36, Iss 4, pp. 968-980.

Bernartzi, S. and R. Thaler (1995). "Myopic Loss Aversion and the Equity Premium Puzzle" *The Quarterly Journal of Economics*, Vol. 110, No. 1, pp. 73-92.

Brunel, J. (2006). "How Sub-Optimal—If at All—Is Goal-Based Asset Allocation?" *Journal of Wealth Management*, Vol. 9, No. 4, pp. 19-34.

Brunel, J. (2011). "Goal-Based Wealth Management in Practice" *Journal of Wealth Management*, Vol 14, No 3, pp. 17-26.

Brunel, J. (2015). *Goals-Based Wealth Management*, Wiley and Sons, Hoboken, NJ.

Chhabra, A. (2005). "Beyond Markowitz: A Comprehensive Wealth Allocation Framework for Individual Investors" *Journal of Wealth Management*, Vol. 7, No. 4, pp. 8-34.

Das, S., H. Markowitz, J. Schied, and M. Statman (2010). "Portfolio Optimization with Mental Accounts" *Journal of Financial and Quantitative Analysis*, Vol. 45, No. 2, pp. 311-334.

Elton, E.J., M.J. Gruber, S.J. Brown, and W.N. Goetzmann. (2009) *Modern Portfolio Theory and Investment Analysis*, John Wiley & Sons.

Friedman, M. and L. Savage (1948). "The Utility Analysis of Choices Involving Risk" *The Journal of Policitcal Economy*, Vol. 56, No. 4, pp. 279-304.

Gonzalez, R. and G. Wu (1999). "On the Shape of the Probability Weighting Function" *Cognitive Psychology*, Vol. 38, pp. 129-166.

Ibbotson, R. G., P. Chen, M. A. Milevsky, X. Zhu (2005). "Human Capital, Asset Allocation, and Life Insurance" *Yale ICF Working Paper No. 05-11.*

Markowitz, H. (1952). "Portfolio Selection" *The Journal of Finance*, Vol. 7, Iss. 1, pp. 77-91.

Markowitz, H. (1952b). "The Utility of Wealth" *The Journal of Political Economy*, Vol. 60, No. 2, pp. 151-158.

Markowitz, H. (1959). *Portfolio Selection: The Efficient Diversification of Investments*, Yale University Press.

Markowitz, H. (2010). "Portfolio Theory: As I Still See It" *Annual Review of Financial Economics*, Vol. 2, pp. 1-23.

Morgenstern, O. and J. von Neumann (1944). *The Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ.

Navarro-Martinez, D., G. Loomes, A. Isoni, D. Butler, and L. Alaoui (2018). "Bounded Rational Expected Utility Theory" *Journal of Risk and Uncertainty*. Vol. 57, pp. 199-223.

Nevins, D. (2004). "Goals-Based Investing: Integrating Traditional and Behavioral Finance" *Journal of Wealth Management*, Vol. 6, No. 4, pp.

Prelec, D. (1998). "The Probability Weighting Function" *Econometrica*, Vol. 66, No. 3, pp. 497-527.

Quiggin, J. (1982). "A Theory of Anticipated Utility" *Journal of Economic Behavior and Organization,* Vol 3, pp. 323-343.

Rabin, M. (2000). "Risk Aversion and Expected Utility Theory: A Calibration Theorem" *Econometrica,* Vol. 68, No. 5, pp. 1281-1292.

Rabin, M. and R. Thaler (2001). "Anomalies: Risk Aversion" *Journal of Economic Perspectives*, Vol. 15, No. 1, pp. 219-232.

Sameulson, P. (1963). "Risk and Uncertainty: A Fallacy of Large Numbers" *Scientia*, Vol 98, pp. 108-113.

Shefrin, H. and M. Statman (2000). "Behavioral Portfolio Theory" *Journal of Financial and Quantitative Analysis*, Vol. 35, No. 2, pp. 127-151.

Statman, M. (2004). "The Diversification Puzzle" *Financial Analysts Journal*, Vol 60, No. 4, pp. 44-53.

Statman, M. (2018). "A Unified Behavioral Finance" *The Journal of Portfolio Management*, Vol. 44, No. 7, pp. 124-134.

Thaler, R. (1985). "Mental Accounting and Consumer Choice" *Marketing Science*, Vol. 4, No. 3, pp. 199-214.

Tversky, A. and D. Kahneman (1992). "Advances in Prospect Theory: Cumulative Representation of Uncertainty" *Journal of Risk and Uncertainty*, Vol 5, pp. 297-323.

Tversky, A., and D. Kahneman (1979). "Prospect Theory: An Analysis of Decision under Risk" *Econometrica*, Vol. 47, No. 2, pp. 263-292.